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CAUCHY INTERNAL WAVE SCATTERING BY DENSITY FIELD INHOMOGENEITIES

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The stationary problem of internal wave (IW) scattering by density field inhomogeneities was considered in a linear formulation in [1] in an unbounded medium with a constant Brunt–Väisäilä frequency. The important role is shown for this mechanism in the IW energy redistribution between different modes. Domains are defined in which the scattered IW amplitude is substantially different from zero. The corresponding nonstationary problem is discussed in this paper.

Let IW characterized by the density $\rho_\Phi(r, t)$ and velocity $U_\Phi(r, t)$ fields exist in a medium. At the time t = 0 local "mixing" (spoilage of the ρ_Φ and U_Φ field distributions) of the medium occur in a domain of space F_1. Neglecting rotation of the earth and the viscosity forces in a Boussinesq approximation, this nonstationary problem has the form

$$L_U\{\rho, \mathbf{U}\} = Q(\mathbf{U}), \quad \mathbf{U}|_{t=0} = \begin{cases} \mathbf{U}_\Phi, & \mathbf{r} \notin D_1, \\ \mathbf{U}_i, & \mathbf{r} \in D_1, \end{cases}$$
$$L_\rho\{\rho, \mathbf{U}\} = \varphi(\rho, \mathbf{U}), \quad \rho|_{t=0} = \begin{cases} \rho_\Phi, & \mathbf{r} \notin D_1, \\ \rho_i, & \mathbf{r} \in D_1, \end{cases} \tag{1}$$

where

$$L_U \equiv \frac{\partial}{\partial t}\Delta\mathbf{U} + \frac{g}{\rho_0}\left[\mathbf{k}\Delta\rho - \nabla\frac{\partial\rho}{\partial z}\right]; \quad L_\rho \equiv \frac{\partial\rho}{\partial t} - \frac{\rho_0}{g}N^2 w; \quad \varphi(\rho, \mathbf{U}) \equiv -\mathbf{U}\nabla\rho;$$
$$Q(\mathbf{U}) \equiv -\operatorname{curl}\operatorname{curl} t\,[(\mathbf{U}\cdot\nabla)\mathbf{U}].$$

The solution of the system (1) can be represented by the sum of two components, one of which describes the problem of the collapse of the intrusion zone in a stratified fluid, and the other the interaction of background IW with this zone. The collapse problem has been investigated well (see, e.g., [2]). It is known [3] that the solution of the collapse problem with viscosity taken into account for large times (the third stage of collapse) is a density field inhomogeneity in the form of a spot of mixed fluid that exists sufficiently long, dissipates extremely slowly at the level of its density. We assume that the geometric size of the domain D_1 and the degree of fluid mixing in it are such that the concluding stage of collapse sets in sufficiently rapidly. Then, following [1], we consider that the domain D that occurs is a density field inhomogeneity that does not change with time and is at rest. Consequently, the problem of background IW interaction with the domain D can be considered as a background IW scattering problem by inhomogeneities of the density field ρ_{i0} with initial conditions. Its solution can also be represented in the form of the sum of two components, one of which described the unperturbed IW field (we consider it known), and the other the intrinsically scattered field characterized by the velocity $U_s(r, t)$ and the density $\rho_s(r, t)$, where $U_s|_{t=0} = 0$ and $\rho_s|_{t=0} = 0$. As in [1], we limit ourselves in this paper to a single scattering approximation (Born approximation) within whose framework U_s and ρ_s satisfy the boundary value problem

$$L_U\{\rho_s, \mathbf{U}_s\} = 0, \quad L_s\{\rho_s, \mathbf{U}_s\} = \varphi(\rho_{i0}, \mathbf{U}_\Phi), \quad \rho_s, \mathbf{U}_s|_{|\mathbf{r}|\to} \to 0. \tag{2}$$

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As before [1], the requirement of smallness of the maximum scattered field amplitude in space and time as compared with the incident wave amplitude can be considered as the necessary condition for applicability of the Born approximation. For instance, for sufficiently large times this condition agrees with the appropriate conditions for the stationary solution [1]. In the general case of arbitrary times, the derivation of constructive constraints on the parameters of the problem (degree of miscibility, size and shape of the scattering volume, etc.) is made difficult because of the complexity of the expression for $\mathbf{U}_S(\rho_S)$. This question should be the subject of a separate investigation and is not examined in this paper. Therefore, assuming the single scattering approximation valid henceforth, we have the following problem for the vertical velocity field component from (2)

$$\frac{\partial^2}{\partial t^2}\Delta w_S + N^2\Delta_h w_S = -N^2\Delta_h(\mathbf{U}_\Phi\cdot\boldsymbol{\beta}),$$

$$w_S|_{t=0} = \left.\frac{\partial w_S}{\partial t}\right|_{t=0} = 0, \quad \boldsymbol{\beta}(r) \equiv -\frac{g}{\rho_0}\frac{\nabla\rho_{i0}}{N^2},$$

whose solution can be expressed in terms of the Green's function

$$w_S(\mathbf{r}, t) = -N^2\int_0^t dt'\int d\mathbf{r}' G(\mathbf{r}-\mathbf{r}', t-t')\Delta_h[\mathbf{U}_\Phi(\mathbf{r}', t')\boldsymbol{\beta}(\mathbf{r}')]$$

($G(\mathbf{r}, t)$ is the Green's function of the internal wave operator [4]). The spatial spectrum is here

$$\tilde{w}_S(\boldsymbol{\Lambda}, t) = \frac{\varkappa^2 N^2}{(2\pi)^3}\int d\boldsymbol{\Lambda}'\int dt'\tilde{\mathbf{U}}_\Phi(\boldsymbol{\Lambda}', t')\tilde{\boldsymbol{\beta}}(\boldsymbol{\Lambda}-\boldsymbol{\Lambda}')\tilde{G}(\boldsymbol{\Lambda}, t-t'), \tag{3}$$

where $\Lambda = \{\boldsymbol{\kappa}, \alpha\}$ is the vector of the wave numbers, and $\tilde{M}$ is the Fourier transform of the function $M(\mathbf{r}, t)$ in the space variable.

Following [1], for the subsequent analysis we make a number of assumptions to simplify the computations, but meanwhile conserve the generality sufficient for many applications:

1) The primary field is a plane monochromatic wave propagating in the negative direction of the coordinate axes at an angle $0 < \alpha_0 < \pi/2$ to the horizontal plane, i.e.,

$$\mathbf{U}_\Phi(\mathbf{r}, t) = \mathbf{A}_0\exp\{i[\mathbf{k}\boldsymbol{\rho} + lz + \omega_0 t]\}, \quad \omega_0 = \frac{k}{\sqrt{k^2+l^2}}N \equiv N\cos\alpha_0;$$

2)
$$\boldsymbol{\beta}(r) = \boldsymbol{\beta}_0 f(\mathbf{r}), \; f(\mathbf{r}) = \begin{cases}1, & \mathbf{r}\in D,\\ 0, & \mathbf{r}\notin D,\end{cases} \quad \boldsymbol{\beta}_0 = \text{const}.$$

Under the assumptions made above, (3) is simplified

$$\tilde{w}_p(\boldsymbol{\Lambda}, t) = \Phi(\boldsymbol{\Lambda})\frac{\theta(t)}{\left(\frac{\varkappa}{\Lambda}N\right)^2-\omega_0^2}\left[\frac{\varkappa}{\Lambda}Ne^{i\omega_0 t} - i\omega_0\sin\left(\frac{\varkappa}{\Lambda}Nt\right) - \frac{\varkappa}{\Lambda}N\cos\left(\frac{\varkappa}{\Lambda}Nt\right)\right],$$

$$\Phi(\boldsymbol{\Lambda}) = -\frac{\varkappa}{\Lambda}N(\mathbf{A}_0\cdot\boldsymbol{\beta}_0)\tilde{f}(\mathbf{k}+\boldsymbol{\varkappa}, l+\alpha), \quad \theta(t) = \begin{cases}1, & t\geqslant 0,\\ 0, & t<0.\end{cases}$$

The relationships

$$\tilde{G}(\boldsymbol{\Lambda}, t) = -\theta(t)\frac{\sin\left(\frac{\varkappa}{\Lambda}Nt\right)}{\varkappa\Lambda N},$$

$$\tilde{\mathbf{U}}_\Phi(\boldsymbol{\Lambda}, t) = (2\pi)^3\mathbf{A}_0\delta(\mathbf{k}+\boldsymbol{\varkappa})\delta(l+\alpha)$$

were used in deriving this formula [$\delta(\mathbf{x})$ is the Dirac delta function].

The change in the density $\rho_S(\mathbf{r}, t)$ associated with the scattering

$$\tilde{\rho}_S(\boldsymbol{\Lambda}, t) = \frac{\rho_0}{g}N^2\Phi(\boldsymbol{\Lambda})\frac{\theta(t)}{\left(\frac{\varkappa}{\Lambda}N\right)^2-\omega_0^2}\left\{\frac{i\omega_0}{\frac{\varkappa}{\Lambda}N}\left[\cos\left(\frac{\varkappa}{\Lambda}Nt\right) - e^{i\omega_0 t}\right] - \sin\left(\frac{\varkappa}{\Lambda}Nt\right)\right\}. \tag{4}$$

is easily calculated from the second equation of the system (2). It is seen from (4) that evolution of the scattered field in time is due to motions of two kinds, as is easily seen by taking the Fourier transform, in time, of (4). Firstly, this is a set of plane waves with the usual dispersion relationship $\omega = \pm N\kappa/\Lambda$ (convergent and divergent waves) and, secondly, motions in the absence of a dispersion relation ("non-wave noise").

We consider the two-dimensional spatial spectrum of the density field $\tilde{\rho}_S(\boldsymbol{\kappa}, z, t)$ at different horizons z in greater detail. To do so we specify the kind of domain D. Let $f(r) = f_1(\boldsymbol{\rho})\theta(L - |z|)$. Since the inverse Fourier transform of α with respect to $\tilde{\rho}_S(\boldsymbol{\Lambda}, t)$ is sufficiently awkward, we limit ourselves to two limit cases $Nt \ll 1$ and $\gg 1$ in analyzing the two-dimensional spectrum $\tilde{\rho}_S(\kappa, z, t)$.

For sufficiently small times, the principal term in the expansion of the two-dimensional spectrum $\tilde{\rho}_S(\boldsymbol{\kappa}, z, t)$ in the parameter $Nt \ll 1$ has the form

$$\tilde{\rho}_S(\varkappa, z, t) = -6P(\varkappa)\theta(L-|z|)e^{ilz}Nt\left[1+\frac{i}{2}\frac{\omega_0}{N}(Nt)-\frac{1}{6}\left(\frac{\omega_0}{N}\right)^2(Nt)^2\right]+$$
$$+P(\varkappa)\frac{(Nt)^3\varkappa}{\varkappa^2+l^2}\Big\{\theta(|z|-L)e^{-|z|\varkappa}[\theta(z)\beta_1\sin(L\beta_2)+\theta(-z)\beta_2\sin(L\beta_1)]+$$
$$+\theta(L-|z|)\left[\varkappa e^{ilz}-\frac{i}{2}\sum_{m=1}^{2}(-1)^m\beta_m e^{(-1)^m(z\varkappa+L\beta_{3-m})}\right]\Big\},$$

where $P(\boldsymbol{\kappa}) = -(\rho_0/6g)N(\mathbf{A_0}\boldsymbol{\beta_0})\tilde{f}_1(\boldsymbol{\kappa} + \mathbf{k})$; $\beta_{1,2} = \ell \pm i\kappa$.

It is seen from the expression obtained that for small times the perturbation occurs in the whole space at once (because of the fluid incompressibility approximation) and grows in proportion to $(Nt)^3$ outside the domain of inhomogeneity. The spectrum amplitude is maximal in the layer occupied by the inhomogeneity, and decreases exponentially above and below this layer, more rapidly as the horizontal wave number increases. In the general case the maximum in the direction $\boldsymbol{\kappa}$ does not agree with the direction of the vector $\mathbf{k}$.

To analyze $\tilde{\rho}_S(\boldsymbol{\kappa}, z, t)$ for long times, we use the method of stationary phase in the large parameter $Nt \gg 1$. In conformity with this method, poles of the integrand and stationary phase points will yield the main contribution to $\tilde{\rho}_S(\boldsymbol{\kappa}, z, t)$. Consequently, it is convenient to separate the asymptotic of the function into two components for large times: $\tilde{\rho}_1(\boldsymbol{\kappa}, z, t)$ (contribution from the poles) and $\tilde{\rho}_2(\boldsymbol{\kappa}, z, t)$ (contribution from the stationary points):

$$\tilde{\rho}_S(\varkappa, z, t) = \tilde{\rho}_1(\varkappa, z, t) + \tilde{\rho}_2(\varkappa, z, t),$$

where

$$\tilde{\rho}_1 = \frac{6}{\varkappa_0}P(\varkappa)e^{i\omega_0 t}\varkappa\Big\{-\theta(|z|-L)\theta(\varkappa_0-b\varkappa)e^{ilz}\times$$
$$\times\left[\theta(z)\frac{\sin(L\eta_1)}{\eta_1}e^{-iz\eta_1}+\theta(-z)\frac{\sin(L\eta_2)}{\eta_2}e^{-iz\eta_2}\right]+$$
$$+\theta(L-|z|)\Bigg[\theta(\varkappa b-\varkappa_0)\,\mathrm{sgn}(z)\sum_{m=1}^{2}(-1)^m\frac{\sin\left(\frac{\eta_{3-m}}{2}(L-|z|)\right)}{\eta_{3-m}}\times$$
$$\times e^{\frac{i}{2}(|z|\eta_m+L\eta_{3-m})\,\mathrm{sgn}(z)}-\theta(\varkappa_0-\varkappa b)\sum_{m=1}^{2}\frac{\sin\left(\frac{\eta_m}{2}(L-(-1)^m z)\right)}{\eta_m}\times$$
$$\times e^{\frac{i}{2}(z\eta_{3-m}+(-1)^m L\eta_m)}\Bigg]\Big\}+i\left(\frac{\omega_0}{N}\right)^2\frac{6}{\varkappa_0}P(\varkappa)\,\mathrm{tg}\,\alpha_0\,\theta(L-|z|)e^{i(lz+\omega_0 t)};$$

$$b\equiv\frac{|z|}{Nt};\quad \eta_{1,2}\equiv l\mp\varkappa\,\mathrm{tg}\,\alpha_0;\quad \varkappa_0\equiv\left(\frac{\omega_0}{N}\right)^2\sqrt{1-\left(\frac{\omega_0}{N}\right)^2};$$

$$\tilde{\rho}_2=\frac{6}{\pi}P(\varkappa)\left(\frac{N}{\omega_0}\right)^2 i\sqrt{\frac{\pi}{2Nt}}\varkappa(\varkappa b)^{-3/4}\sum_{k=1}^{2}\frac{(c_k)^{9/4}}{\left[\varkappa b\left(\frac{N}{\omega_0}\right)^2-c_k\right]\sqrt{|3\varkappa b-2c_k|}}\times$$
$$\times\sum_{m=1}^{2}\Bigg\{\left[\theta(z)\frac{\sin(L(l+(-1)^m\alpha_k))}{l+(-1)^m\alpha_k}+\theta(-z)\frac{\sin(L(l-(-1)^m\alpha_k))}{l-(-1)^m\alpha_k}\right]\times$$
$$\times\left[\frac{\omega_0}{N}-(-1)^m\sqrt{\frac{\varkappa b}{c_k}}\right]\Bigg\}e^{(-1)^{m-1}iS(\alpha_k)};$$

$$b = \text{const};\quad \varkappa b \neq \varkappa_0;\quad \alpha_k^2 = \frac{\varkappa}{b}(c_k - \varkappa b);\quad \cos\gamma = -\frac{3\sqrt{3}}{2}\varkappa b;$$

$$c_1 = \frac{2}{\sqrt{3}}\cos\frac{\gamma}{3};\quad c_2 = -\frac{2}{\sqrt{3}}\cos\left(\frac{\gamma+\pi}{3}\right);\quad S(\alpha_k) = \sqrt{\frac{\varkappa b}{c_k}}\,Nt + |z|\,\alpha_k - (-1)^k\frac{\pi}{4}.$$

It is seen from the expression obtained that for long times when $|\tilde{\rho}_1| \gg |\tilde{\rho}_2|$ the scattered field frequency tends to the incident field frequency. The spectrum maximum here occurs at $\kappa \approx k$ and corresponds, in the propagation direction, to the angles $\psi = \pm\alpha_0$, $\psi = \pm\alpha_0 + \pi$. As the time grows further the spectrum maximum is increased and the function $\tilde{\rho}_S(\boldsymbol{\kappa}, z, t)$ tends to the stationary solution of the problem [1]. The spectrum has an upper bound at each time on a given horizon, and its upper boundary increases as time elapses (i.e., as time lapses all the shorter waves arrive at the given horizon). In the layer occupied by the inhomogeneity a whole set of horizontal wave numbers exists at once. The appearance of the component $\tilde{\rho}_2$ in the asymptotic of the spectrum is due exclusively to the presence of initial conditions. Consequently, the component of the scattered field spectrum depends on the time complexly. As time elapses the frequency of oscillation of the component $\tilde{\rho}_2$ changes continuously for a given wave number (a given wavelength), to form two groups of waves with the frequencies ~N and $\sim N\sqrt{\kappa b}$. Both these wave groups decrease as t and κ grow, where the vibrations with frequency $\omega \sim N\sqrt{\kappa b}$ "die out" more slowly with time than those with the frequency $\omega \sim N$. It is interesting to note that as $\omega \sim N\sqrt{\kappa b}$ the vertical wave number $\alpha \approx \sqrt{\kappa/b}$ can also be sufficiently large. This circumstance can apparently be related to the mechanism for the origination of a long-lived fine vertical structure: its formation occurs under the effect of the scattered waves, where the fine vertical structure is formed on all the higher (lower) horizons as time lapses.

Thus, the IW scattering pattern by a density field inhomogeneity is represented as follows (for definiteness, we speak about the density field parameters). At the time of the appearance of inhomogeneities in the medium, density field perturbations occur that damp out exponentially with distance from the scattering volume. Far from the inhomogeneities, the perturbation amplitude is isotropic in space and grows in proportion to $(Nt)^3$. As time passes, the scattered field acquires the nature of a wave, where long-wave vibrations appear first at a given horizon, and then shorter and shorter waves reach it. The two-dimensional spatial spectrum of the density field is deformed continuously: the oscillation amplitude damps out with time while their distribution becomes anisotropic in space, being concentrated near the directions $\psi \approx \pm\alpha_0$ and $\psi \approx \pm\alpha_0 + \pi$ as Nt grows. In the limit as $Nt \to \infty$ the scattered field emerges in the stationary regime. Its amplitude is substantially nonzero near the direction ψ and decreases with distance from the inhomogeneity. The time dependence here has the form $e^{i\omega_0 t}$ ("standing" wave).

Therefore, the nonstationary solution of the IW scattering problem by localized density field inhomogeneities affords the possibility of tracking the energy redistribution in the IW spectrum in time, and thereby expanded the representation of this mechanism as compared with [1]. Unfortunately, a more detailed investigation of the nonstationary solutions is impossible in even the simplest formulation that we proposed because of the vastness of the calculations that occur. Consequently, a further analytical analysis of both the stationary and nonstationary solutions of the IW scattering problem within the framework of the model considered is inexpedient in our opinion. Most promising by clarification of the scattered field "geometry" and its evolution in time is the passage to computations on an electronic computer with the use of the asymptotic expressions. Such computations will assist in obtaining a formulation and solution of one of the most interesting problems associated with IW scattering: clarification of the zones of increase in the local IW instability and their degeneration into turbulence. The important theoretical aspect of the solution of this problem is the inclusion of viscosity in the model we considered and taking account of the spatial variability of the Brunt–Väisäilä frequency (taking account of the fine structure).

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